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Relationship between squeezing and entangled state transformations

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Abstract

We show that c-number dilation transform in the Einstein–Podolsky–Rosen (EPR) entangled state, i.e. $|\eta_1, \eta_2\rangle \rightarrow |\eta_1, \eta_2/\mu\rangle$ (or $|\eta_1, \eta_2\rangle \rightarrow |\eta_1/\mu, \eta_2\rangle$), maps onto a kind of one-sided two-mode squeezing operator $\exp\left\{\mathrm{i}\frac{\lambda}{2}(P_1 + P_2)(Q_1 + Q_2) - \frac{\lambda}{2}\right\}$ $\left(\text{or} \exp\left\{\mathrm{i}\frac{\lambda}{2}(P_1 - P_2)(Q_1 - Q_2) - \frac{\lambda}{2}\right\}\right)$. Using the IWOP technique, we derive their normally ordered form and construct the corresponding squeezed states. In doing so, some new relationship between squeezing and entangled state transformation is revealed. The dynamic Hamiltonian for such a kind of squeezing evolution is derived. The properties and application of the one-sided squeezed state are briefly discussed. These states can also be obtained with the use of a beam splitter.

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1. Introduction

Squeezed states of a light is an important topic in quantum optics [1, 2] and will be more widely used in quantum detection, because the quantum fluctuation of one quadrature of a light field in a squeezed state is less than that in a coherent state. The two-mode squeezed states will have potential uses in quantum communication because the idler-mode photon and signal-mode photon in a two-mode squeezed state are entangled in the frequency domain (a quantum entanglement phenomenon). One may put it in another way, that the correlations between idler mode and signal mode give rise to two-mode squeezing [3]. This is evidence that in two-mode systems there is a richer variety of quantum phenomena demonstrating quantum correlations between the modes. For two-mode optical fields, the quadrature operators characteristic of quantum fluctuations of various states are [1]

$$\frac{a_1 + a_2 + a_1^\dagger + a_2^\dagger}{2^{3/2}} \equiv X \qquad \frac{a_1 + a_2 - a_1^\dagger - a_2^\dagger}{2^{3/2}\mathrm{i}} \equiv P \tag{1}$$

satisfying the commutative relation

$$[X, P] = \mathrm{i}/2. \tag{2}$$

If one introduces

$$\frac{a_i + a_i^\dagger}{\sqrt{2}} = Q_i \qquad \frac{a_i - a_i^\dagger}{\sqrt{2}\mathrm{i}} = P_i \qquad i = 1, 2 \tag{3}$$

then

$$X = (Q_1 + Q_2)/2 \qquad P = (P_1 + P_2)/2. \tag{4}$$

The two-mode squeezing operator is [1]

$$S'' = \exp\left[\lambda\left(a_1^\dagger a_2^\dagger - a_1 a_2\right)\right] = \exp[-\mathrm{i}\lambda(P_2 Q_1 + Q_2 P_1)] \tag{5}$$

which squeezes

$$S'' X S''^{-1} = \mathrm{e}^{\lambda} X \qquad S'' P S''^{-1} = \mathrm{e}^{-\lambda} P. \tag{6}$$

Based on the idea of quantum entanglement proposed by Einstein–Podolsky–Rosen (EPR) [4], we have revealed in [5] that the two-mode squeezing operator actually squeezes the EPR entangled state $|\eta\rangle$, i.e.

$$S''|\eta\rangle = \frac{1}{\mu}\left|\frac{\eta}{\mu}\right\rangle \qquad \mu = \mathrm{e}^{\lambda} \tag{7}$$

where $|\eta\rangle$ is defined as [6, 7]

$$|\eta\rangle = \exp\left[-\tfrac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right]|00\rangle. \tag{8}$$

$\eta = \eta_1 + \mathrm{i}\eta_2$ is a complex number, $|00\rangle$ is the two-mode vacuum state. $|\eta\rangle$ obeys the eigenvector equations

$$(Q_1 - Q_2)|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle \qquad (P_1 + P_2)|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle \tag{9}$$

namely, $|\eta\rangle$ is the common eigenvector of $P_1 + P_2$ and $Q_1 - Q_2$. The motivation of paper [5] stemmed from two observations: (1) the quadrature operator $(P_1 + P_2)/2$ is one of the object operators discussed in the paper of Einstein–Podolsky–Rosen, who used the fact that the total momentum of the two particles, $P_1 + P_2$, commutes with their relative coordinate $Q_1 - Q_2$ to elucidate a mysterious correlation between the two particles [4], (2) the two-mode squeezing operator S'' causes quantum entanglement between the two modes of squeezed light as demonstrated in a parametric down conversion process. Thus, there must exist an intrinsic relation between S'' and the EPR-entangled $|\eta\rangle$. Because $|\eta\rangle$ makes up a complete and orthonormal representation [5]

$$\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta| = 1 \tag{10}$$

$$\langle\eta'|\eta\rangle = \pi\delta(\eta - \eta')\delta(\eta^* - \eta'^*) = \pi\delta(\eta_1' - \eta_1)\delta(\eta_2' - \eta_2) \equiv \pi\delta^{(2)}(\eta - \eta') \tag{11}$$

we have

$$S'' = \frac{1}{\mu}\int \frac{\mathrm{d}^2\eta}{\pi}\left|\frac{\eta}{\mu}\right\rangle\langle\eta| \qquad \mu = \mathrm{e}^{\lambda} \tag{12}$$

which means that the two-mode squeezing operator just has its natural representation in the $|\eta\rangle$ representation. The physical meaning in (12) is twofold: (1) it directly relates the EPR state with the two-mode squeezing, we now understand more deeply that the two-mode squeezed state has its idler mode and signal mode entangled with each other, (2) equation (12) shows that the quantum mechanical image of a classical dilation $\eta \to \frac{\eta}{\mu}$ in the EPR state $|\eta\rangle$ is just the two-mode squeezing operator. Like $q \to q/\mu$ in a coordinate eigenstate $|q\rangle$ mapping onto the single-mode squeezing operator, we see that the ideal EPR state $|\eta\rangle$ is on the same footing as the state $|q\rangle$, so far as the squeezing is concerned. The state $|\eta\rangle$, as a fundamental basis vector, is indispensable for continuous variable entangled states.

If we rewrite $|\eta\rangle$ as

$$|\eta\rangle = \exp\left[-\tfrac{1}{2}|\eta|^2 + \eta_1\left(a_1^\dagger - a_2^\dagger\right) + \mathrm{i}\eta_2\left(a_1^\dagger + a_2^\dagger\right) + a_1^\dagger a_2^\dagger\right]|00\rangle \equiv |\eta_1, \eta_2\rangle \tag{13}$$

then an interesting question naturally arises: what are the squeezing operators which respectively squeezes $|\eta_1, \eta_2\rangle \to |\eta_1, \eta_2/\mu\rangle$ and $|\eta_1, \eta_2\rangle \to |\eta_1/\mu, \eta_2\rangle$? In this work we shall reveal that there exists another kind of two-mode squeezing operator,

$$S \equiv \mathrm{e}^{\mathrm{i}\frac{\lambda}{2}(P_1+P_2)(Q_1+Q_2)-\frac{\lambda}{2}} = \exp\left[\frac{\lambda}{4}\left(a_1^2 - a_1^{\dagger 2} + a_2^2 - a_2^{\dagger 2} + 2a_1a_2 - 2a_1^\dagger a_2^\dagger\right)\right] \tag{14}$$

which plays the role of asymmetric shrink transform

$$S^{-1}|\eta_1, \eta_2\rangle \to \frac{1}{\sqrt{\mu}}|\eta_1, \eta_2/\mu\rangle \tag{15}$$

and the operator $S' \equiv \mathrm{e}^{\mathrm{i}\frac{\lambda}{2}(P_1-P_2)(Q_1-Q_2)-\frac{\lambda}{2}} = \exp\left[\frac{\lambda}{4}\left(a_1^2 - a_1^{\dagger 2} + a_2^2 - a_2^{\dagger 2} - 2a_1a_2 + 2a_1^\dagger a_2^\dagger\right)\right]$ makes the transform $S'^{-1}\,|\eta_1, \eta_2\rangle = \frac{1}{\sqrt{\mu}}\,|\eta_1/\mu, \eta_2\rangle$. We name them one-sided squeezing operators and shall prove their roles in the following sections. After deducing their normally ordered forms, we also construct the corresponding squeezed states. We shall also briefly compare these more general squeezed states with the conventional two-mode squeezed state and discuss how to produce them dynamically. The Wigner functions of these one-sided squeezed states will also be derived by virtue of the $|\eta\rangle$ representation. In doing so, some new relationship between squeezing and the asymmetric shrink transform of the entangled state is revealed. The dynamic Hamiltonian for generating one-sided squeezed states is derived, and its generation with the use of a beam splitter is discussed.

2. The squeezing $|\eta_1, \eta_2\rangle \to |\eta_1, \eta_2/\mu\rangle$

The Schmidt decomposition of $|\eta\rangle$ is [8]

$$|\eta = \eta_1 + \mathrm{i}\eta_2\rangle = \mathrm{e}^{-\mathrm{i}\eta_2\eta_1}\int_{-\infty}^{\infty}\mathrm{d}q\;|q\rangle_1 \otimes |q - \sqrt{2}\eta_1\rangle_2\,\mathrm{e}^{\mathrm{i}\sqrt{2}\eta_2 q} \tag{16}$$

where

$$|q\rangle_i = \pi^{-\frac{1}{4}}\exp\left[-\tfrac{1}{2}q^2 + \sqrt{2}qa_i^\dagger - \tfrac{1}{2}a_i^{\dagger 2}\right]|0\rangle_i \tag{17}$$

is the coordinate eigenstate, $Q_i|q\rangle_i = q|q\rangle_i$, the subscript $i = 1$ ($i = 2$) denotes the a_1-mode (a_2-mode) Fock space in which the coordinate eigenvector $|q\rangle_i$ is defined. The $|\eta\rangle$ state can also be Schmidt-decomposed as

$$|\eta\rangle = \mathrm{e}^{-\mathrm{i}\eta_1\eta_2}\int_{-\infty}^{\infty}\mathrm{d}p\,|p + \sqrt{2}\eta_2\rangle_1 \otimes |-p\rangle_2\,\mathrm{e}^{-\mathrm{i}\sqrt{2}\eta_1 p} \tag{18}$$

where $|p\rangle_i$ is the momentum eigenvector

$$|p\rangle_i = \pi^{-\frac{1}{4}}\exp\left[-\tfrac{1}{2}p^2 + \mathrm{i}\sqrt{2}pa_i^\dagger + \tfrac{1}{2}a_i^{\dagger 2}\right]|0\rangle_i. \tag{19}$$

Using equation (16) we easily obtain

$$(Q_1 + Q_2)|\eta\rangle = \int_{-\infty}^{\infty}\mathrm{d}x(2q - \sqrt{2}\eta_1)|q\rangle_1 \otimes |q - \sqrt{2}\eta_1\rangle_2\,\mathrm{e}^{\mathrm{i}\eta_2(2q-\sqrt{2}\eta_1)/\sqrt{2}} = -\mathrm{i}\sqrt{2}\frac{\partial}{\partial\eta_2}|\eta\rangle. \tag{20}$$

On the other hand, using equation (18) we have

$$(P_1 - P_2)|\eta\rangle = \int_{-\infty}^{\infty} \mathrm{d}p\,(2p + \sqrt{2}\eta_2)|p + \sqrt{2}\eta_2\rangle_1 \otimes |-p\rangle_2\,\mathrm{e}^{-\mathrm{i}\eta_1(2p+\sqrt{2}\eta_2)/\sqrt{2}} = \mathrm{i}\sqrt{2}\frac{\partial}{\partial\eta_1}|\eta\rangle. \tag{21}$$

From (20) and (21) we see that in the $\langle\eta|$ representation

$$(Q_1 + Q_2) \to \mathrm{i}\sqrt{2}\frac{\partial}{\partial\eta_2} \qquad (P_1 - P_2) \to -\mathrm{i}\sqrt{2}\frac{\partial}{\partial\eta_1}.$$

Combining equation (20) with equation (9) we have

$$\langle\eta|\frac{1}{2}(P_1 + P_2)(Q_1 + Q_2) = \mathrm{i}\eta_2\frac{\partial}{\partial\eta_2}\langle\eta|. \tag{22}$$

Let $\eta_2 \equiv \mathrm{e}^y$, from (22) we derive

$$\langle\eta|\frac{1}{2}(P_1 + P_2)(Q_1 + Q_2) = \mathrm{i}\,\mathrm{e}^y\frac{\partial y}{\partial\eta_2}\frac{\partial}{\partial y}\langle\eta_1, \eta_2 = \mathrm{e}^y| = \mathrm{i}\frac{\partial}{\partial y}\langle\eta_1, \eta_2 = \mathrm{e}^y|. \tag{23}$$

It then follows from (23) and $\mathrm{e}^{-\lambda\frac{\partial}{\partial y}}f(y) = f(y-\lambda)$ that

$$\langle\eta|\,\mathrm{e}^{\mathrm{i}\lambda(P_1+P_2)(Q_1+Q_2)/2} = \mathrm{e}^{-\lambda\frac{\partial}{\partial y}}\langle\eta_1, \eta_2 = \mathrm{e}^y| = \langle\eta_1, \mathrm{e}^{y-\lambda}| = \langle\eta_1, \mathrm{e}^{-\lambda}\eta_2| \tag{24}$$

which means that the unitary operator $S \equiv \mathrm{e}^{\mathrm{i}\frac{\lambda}{2}(P_1+P_2)(Q_1+Q_2)-\frac{\lambda}{2}}$ squeezes $\langle\eta|$ as

$$\langle\eta|S = 1/\sqrt{\mu}\langle\eta_1, \eta_2/\mu|. \tag{25}$$

Now using the completeness relation (10), we know the $\langle\eta|$ representation of S,

$$S = \mathrm{e}^{-\lambda/2}\int\frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta_1, \mathrm{e}^{-\lambda}\eta_2| = \mathrm{e}^{\lambda/2}\int\frac{\mathrm{d}^2\eta}{\pi}|\eta_1, \mathrm{e}^{\lambda}\eta_2\rangle\langle\eta|. \tag{26}$$

Equation (26) shows that the quantum mechanical image of a classical dilation $\eta_2 \to \mathrm{e}^{\lambda}\eta_2$ in the EPR state $|\eta\rangle$ is just the one-sided squeezing operator S.

3. The normally ordered form of S

Using (13) and the normally ordered expansion of the two-mode vacuum state

$$|00\rangle\langle 00| =: \exp\left[-a_1^\dagger a_1 - a_2^\dagger a_2\right]:$$

as well as the technique of integration within an ordered product (IWOP) of operators [9–11], we can perform the integration in (26) and obtain its normally ordered expansion

$$\begin{aligned}
S = \mathrm{e}^{-\lambda/2}\int\frac{\mathrm{d}^2\eta}{\pi} :\exp&\left\{-\eta_1^2 - \frac{1}{2}\eta_2^2(1+\mathrm{e}^{-2\lambda}) + \eta_1\left(a_1^\dagger - a_2^\dagger\right) + \mathrm{i}\eta_2\left(a_1^\dagger + a_2^\dagger\right)\right.\\
&\left.+ a_1^\dagger a_2^\dagger + \eta_1(a_1 - a_2) - \mathrm{i}\,\mathrm{e}^{-\lambda}\eta_2(a_1 + a_2) + a_1a_2 - a_1^\dagger a_1 - a_2^\dagger a_2\right\}:\\
= \mathrm{e}^{-\lambda/2}\sqrt{\frac{2}{1+\mathrm{e}^{-2\lambda}}}:\exp&\left\{\frac{1}{4}\left(a_1^\dagger - a_2^\dagger + a_1 - a_2\right)^2 + a_1^\dagger a_2^\dagger + a_1a_2 - a_1^\dagger a_1 - a_2^\dagger a_2\right.\\
&\left.- \frac{1}{2(1+\mathrm{e}^{-2\lambda})}\left[a_1^\dagger + a_2^\dagger - \mathrm{e}^{-\lambda}(a_1+a_2)\right]^2\right\}:\\
= \mathrm{sech}^{1/2}\lambda :\exp&\left\{-\frac{\tanh\lambda}{4}\left[\left(a_1^\dagger + a_2^\dagger\right)^2 - (a_1+a_2)^2\right]\right.\\
&\left.+ \frac{1}{2}(\mathrm{sech}\,\lambda - 1)\left(a_1^\dagger + a_2^\dagger\right)(a_1+a_2)\right\}:.
\end{aligned} \tag{27}$$

where $\tanh\lambda \equiv \frac{\mu^2-1}{\mu^2+1}$, $\operatorname{sech}\lambda = \frac{2\mu}{\mu^2+1}$. With the aid of the operator identity

$$\exp\left[f\left(a_1^\dagger + a_2^\dagger\right)(a_1 + a_2)\right] = :\exp\left[\tfrac{1}{2}(\mathrm{e}^{2f} - 1)\left(a_1^\dagger + a_2^\dagger\right)(a_1 + a_2)\right]: \tag{28}$$

we can further write (27) as

$$S = \operatorname{sech}^{1/2}\lambda\, \mathrm{e}^{-\frac{\tanh\lambda}{4}(a_1^\dagger+a_2^\dagger)^2}\, \mathrm{e}^{\frac{1}{2}(a_1^\dagger+a_2^\dagger)(a_1+a_2)\ln\operatorname{sech}\lambda}\, \mathrm{e}^{\frac{\tanh\lambda}{4}(a_1+a_2)^2}. \tag{29}$$

This is a new squeezing operator. Operating (29) on the vacuum state $|00\rangle$ yields a new two-mode squeezed vacuum state

$$S|00\rangle = \operatorname{sech}^{1/2}\lambda \exp\left\{-\frac{\tanh\lambda}{4}\left(a_1^\dagger + a_2^\dagger\right)^2\right\}|00\rangle \tag{30}$$

or

$$S|00\rangle = \mathrm{e}^{-\lambda/2}\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta_1, \mathrm{e}^{-\lambda}\eta_2|00\rangle = \mathrm{e}^{-\lambda/2}\int\frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\, \mathrm{e}^{-(\eta_1^2+\eta_2^2\mathrm{e}^{-2\lambda})/2}. \tag{31}$$

One may compare equation (30) with the usual two-mode squeezed state

$$\begin{gathered} S''^{-1}|00\rangle = \int\frac{\mathrm{d}^2\eta}{\pi\mu}|\eta\rangle\, \mathrm{e}^{-|\eta|^2/2\mu^2} = \operatorname{sech}\lambda\, \mathrm{e}^{-a_1^\dagger a_2^\dagger\tanh\lambda}|00\rangle = \operatorname{sech}\lambda\sum_{n=0}(-\tanh\lambda)^n|n,n\rangle \\ |n,n\rangle \equiv |n\rangle_1|n\rangle_2 \qquad |n\rangle_1 = \frac{a_1^{\dagger n}}{\sqrt{n!}}|0\rangle_1 \end{gathered} \tag{32}$$

and see their difference, that is, in equation (30) the operators $a_1^{\dagger 2}, a_2^{\dagger 2}$ and $a_1^\dagger a_2^\dagger$ coexist. Expanding the exponential in (30), we have

$$\begin{aligned} S|00\rangle &= \operatorname{sech}^{1/2}\lambda\sum_{n=0}^{\infty}\left(-\frac{\tanh\lambda}{4}\right)^n\frac{1}{n!}\left(a_1^\dagger + a_2^\dagger\right)^{2n}\}|00\rangle \\ &= \operatorname{sech}^{1/2}\lambda\sum_{n=0}^{\infty}\left(-\frac{\tanh\lambda}{2}\right)^n\sum_{l=0}^{2n}\frac{1}{\sqrt{l!(2n-l)!}}|l, 2n-l\rangle. \end{aligned}$$

Thus, we see that the photon number distribution of the state $S|00\rangle$ is different from that of the state $S''^{-1}|00\rangle$, the latter contains only the twin number state $|n,n\rangle$. By introducing the Noh–Fougères–Mandel [12, 13] operational phase operator $\sqrt{\frac{a_1-a_2^\dagger}{a_2-a_1^\dagger}}$ and noting $\left[a_1-a_2^\dagger, a_2-a_1^\dagger\right] = 0$, and

$$\left(a_1 - a_2^\dagger\right)|\eta\rangle = \eta|\eta\rangle \qquad \left(a_2 - a_1^\dagger\right)|\eta\rangle = \eta^*|\eta\rangle$$

we see that its eigenvector is also the entangled state $|\eta = |\eta|\mathrm{e}^{\mathrm{i}\varphi}\rangle$ with the eigenvalue being $\mathrm{e}^{\mathrm{i}\varphi}$,

$$\sqrt{\frac{a_1 - a_2^\dagger}{a_2 - a_1^\dagger}}|\eta\rangle = \mathrm{e}^{\mathrm{i}\varphi}|\eta\rangle \qquad \varphi = \arctan\left(\frac{\eta_2}{\eta_1}\right).$$

The phase property of our new state $S|00\rangle$ is also different from that of the usual two-mode squeezed state, because from (31) we see

$$\sqrt{\frac{a_1 - a_2^\dagger}{a_2 - a_1^\dagger}}S|00\rangle = \mathrm{e}^{-\lambda}\int\frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\, \mathrm{e}^{-(\eta_1^2+\eta_2^2\mathrm{e}^{-2\lambda})/2}\exp\left[\mathrm{i}\arctan\left(\frac{\eta_2}{\eta_1}\right)\right].$$

This integration is different from

$$\sqrt{\frac{a_1 - a_2^\dagger}{a_2 - a_1^\dagger}} S''^{-1}|00\rangle = \mathrm{e}^{-\lambda} \int \frac{\mathrm{d}^2\eta}{\pi} |\eta\rangle\, \mathrm{e}^{-|\eta^2|\mathrm{e}^{-2\lambda}/2}\, \mathrm{e}^{\mathrm{i}\varphi}.$$

Mandel *et al* [3, 12, 13] have shown that in the phase measurement scheme using an eight-port interferometer with four input photon modes, in the strong local oscillator limit and for some particular reference phase, what the two detector pairs of the eight-port interferometer measure actually are the two-mode operators $(Q_1 - Q_2)$ and$(P_1 + P_2)$, which correspond to the difference in the photon numbers at the output, so the squeezing of either $(Q_1 - Q_2)$ or $(P_1 + P_2)$ is worth discussing from both practical and theoretical point of views. The variation of the photon-number difference may indirectly cause or affect the squeezing.

Now we introduce the state the common eigenvector $|\xi\rangle$ [7] of another pair of EPR commutative operators $Q_1 + Q_2$ and $P_1 - P_2$; its explicit form in two-mode Fock space is

$$|\xi\rangle = \exp\left[-\tfrac{1}{2}|\xi|^2 + \xi a_1^\dagger + \xi^* a_2^\dagger - a_1^\dagger a_2^\dagger\right]|00\rangle \equiv |\xi_1, \xi_2\rangle \tag{33}$$

where $\xi = \xi_1 + \mathrm{i}\xi_2$

$$(Q_1 + Q_2)|\xi\rangle = \sqrt{2}\xi_1|\xi\rangle \qquad (P_1 - P_2)|\xi\rangle = \sqrt{2}\xi_2|\xi\rangle. \tag{34}$$

is also complete

$$\int \frac{\mathrm{d}^2\xi}{\pi} |\xi\rangle\langle\xi| = 1.$$

The overlap between $\langle\eta|$ and $|\xi\rangle$ is [8]

$$\langle\eta|\xi\rangle = \tfrac{1}{2}\exp[\mathrm{i}(\eta_1\xi_2 - \eta_2\xi_1). \tag{35}$$

We can prove that in the $\langle\xi|$ representation

$$\begin{aligned}
S &= \mathrm{e}^{-\lambda/2} \int \frac{\mathrm{d}^2\eta}{\pi} |\eta\rangle\langle\eta_1, \mathrm{e}^{-\lambda}\eta_2| \int \frac{\mathrm{d}^2\xi}{\pi} |\xi\rangle\langle\xi| \\
&= \mathrm{e}^{-\lambda/2} \int \frac{\mathrm{d}^2\xi}{\pi} \int \frac{\mathrm{d}^2\eta}{2\pi} \exp\left[-\frac{1}{2}|\eta|^2 + \eta_1\left(a_1^\dagger - a_2^\dagger + \mathrm{i}\xi_2\right)\right. \\
&\quad \left. + \mathrm{i}\eta_2\left(a_1^\dagger + a_2^\dagger - \mathrm{e}^{-\lambda}\xi_1\right) + a_1^\dagger a_2^\dagger\right] |00\rangle\langle\xi| \\
&= \mathrm{e}^{-\lambda/2} \int \frac{\mathrm{d}^2\xi}{\pi} \exp\left[-\frac{1}{2}\mathrm{e}^{-2\lambda}\xi_1^2 - \frac{1}{2}\xi_2^2 + \mathrm{i}\xi_2\left(a_1^\dagger - a_2^\dagger\right)\right. \\
&\quad \left. + \mathrm{e}^{-\lambda}\xi_1\left(a_1^\dagger + a_2^\dagger\right) - a_1^\dagger a_2^\dagger\right] |00\rangle\langle\xi| \\
&= \mathrm{e}^{-\lambda/2} \int \frac{\mathrm{d}^2\xi}{\pi} |\mathrm{e}^{-\lambda}\xi_1, \xi_2\rangle\langle\xi|
\end{aligned} \tag{36}$$

namely,

$$S|\xi\rangle = \mathrm{e}^{-\frac{\lambda}{2}}|\mathrm{e}^{-\lambda}\xi_1, \xi_2\rangle. \tag{37}$$

The quantum fluctuation of quadratures in the new squeezed states is the same as that in the usual squeezed state. To see this, using the operator identity

$$\mathrm{e}^A B\, \mathrm{e}^{-A} = B + [A, B] + \tfrac{1}{2!}[A, [A, B]] + \tfrac{1}{3!}[A, [A, [A, B]]] + \cdots$$

we obtain

$$SPS^{-1} = \mathrm{e}^{-\lambda}P \qquad SXS^{-1} = \mathrm{e}^{\lambda}X \tag{38}$$

which is in agreement with (37) and (26). It then follows that $\langle 0|S^{-1}XS|0\rangle = 0$, $\langle 0|S^{-1}PS|0\rangle = 0$. So the quantum fluctuations of quadrature operators in the state $S|0\rangle$ are $(\triangle X)^2 = \frac{1}{4\mu^2}$, $(\triangle P)^2 = \frac{\mu^2}{4}$, which are the same as in the ordinary two-mode squeezed vacuum state.

4. The Hamiltonian for generating the squeezing

We now present the dynamic Hamiltonian for generating such a squeezing evolution. Let the squeezing parameter $\mu = \mathrm{e}^{\lambda}$ in equation (26) (or (27)) be time-dependent $\mu(t) = \mathrm{e}^{\lambda(t)}$, we seek the interaction Hamiltonian which can generate the continuous squeezing transform $|\eta\rangle \rightarrow |\eta_1, \mathrm{e}^{\lambda}\eta_2\rangle$. For this purpose, we differentiate (29) with respect to t and use (21) to obtain

$$\mathrm{i}\frac{\partial S(\mu(t))}{\partial t} = \mathrm{i}\frac{\partial \lambda(t)}{\partial t}\left((a_1 + a_2)^2 - \left(a_1^{\dagger} + a_2^{\dagger}\right)^2\right)S(\mu(t)). \tag{39}$$

Comparing (39) with the standard form for the Schrödinger equation $\mathrm{i}\frac{\partial S(\mu(t))}{\partial t} = H(t)S(\mu(t))$ in an interaction picture, we know the Hamiltonian is

$$H(t) = \frac{\partial \lambda(t)}{\partial t}\frac{\mathrm{i}}{4}\left[(a_1 + a_2)^2 - \left(a_1^{\dagger} + a_2^{\dagger}\right)^2\right] = -\frac{\partial \lambda(t)}{\partial t}\frac{1}{2}[(P_1 + P_2)(Q_1 + Q_2) + \mathrm{i}]. \tag{40}$$

Such a dynamic mechanism may happen in a second-harmonic generation or parametric down-conversion process. To realize it, one needs two incoming optical fields (E_1 and E_2) and a nonlinear crystal for which the polarization $\wp$ is not only a linear function of the two incoming electric fields $E_1 + E_2$ but also involves quadratic (and even higher) powers of $E_1 + E_2$, that is

$$\wp = \chi^{(1)}(E_1 + E_2) + \chi^{(2)}(E_1 + E_2)^2 + \cdots$$

where

$$E_i(x, t) \sim a_i\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} + a_{\mathrm{i}}^{\dagger}\,\mathrm{e}^{-\mathrm{i}(kx-\omega t)}.$$

5. The squeezing corresponding to the transform $|\eta_1, \eta_2\rangle \rightarrow |\eta_1/\mu, \eta_2\rangle$

Similarly, from equations (9) and (21) we have

$$\langle\eta|(Q_1 - Q_2)(P_1 - P_2)/2 = -\mathrm{i}\eta_1\frac{\partial}{\partial\eta_1}\langle\eta| \tag{41}$$

then letting $\eta_1 = \mathrm{e}^x$, we see

$$\langle\eta|(Q_1 - Q_2)(P_1 - P_2)/2 = -\mathrm{i}\,\mathrm{e}^x\frac{\partial x}{\partial\eta_1}\frac{\partial}{\partial x}\langle\eta_1 = \mathrm{e}^x, \eta_2| = -\mathrm{i}\frac{\partial}{\partial x}\langle\eta_1 = \mathrm{e}^x, \eta_2|. \tag{42}$$

It then follows that

$$\begin{aligned} S' &\equiv \mathrm{e}^{-\mathrm{i}\frac{\lambda}{2}(Q_1-Q_2)(P_1-P_2)-\frac{\lambda}{2}} = \mathrm{e}^{-\frac{\lambda}{2}}\int\frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\left(\mathrm{e}^{-\lambda\frac{\partial}{\partial x}}\langle\eta_1 = \mathrm{e}^x, \eta_2|\right) \\ &= \mathrm{e}^{-\frac{\lambda}{2}}\int\frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\mathrm{e}^{-\lambda}\eta_1, \eta_2| \end{aligned} \tag{43}$$

or

$$S' = \mathrm{e}^{\frac{\lambda}{2}}\int\frac{\mathrm{d}^2\eta}{\pi}|\mathrm{e}^{\lambda}\eta_1, \eta_2\rangle\langle\eta| = \mathrm{e}^{-\frac{\lambda}{2}}\int\frac{\mathrm{d}^2\xi}{\pi}|\xi_1, \mathrm{e}^{-\lambda}\xi_2\rangle\langle\xi|. \tag{44}$$

Using the IWOP technique, we perform the integration in (40) and obtain

$$S' = \frac{1}{\sqrt{\cosh\lambda}} \exp\left\{-\frac{\tanh\lambda}{4}\left(a_1^\dagger - a_2^\dagger\right)^2\right\} \exp\left\{-\frac{1}{2}\left(a_1^\dagger - a_2^\dagger\right)(a_1 - a_2) \ln\cosh\lambda\right\} \\ \times \exp\left\{\frac{\tanh\lambda}{4}(a_1 - a_2)^2\right\}. \tag{45}$$

The corresponding squeezed vacuum state is

$$S'|00\rangle = \frac{1}{\sqrt{\cosh\lambda}} \exp\left\{-\frac{\tanh\lambda}{4}\left(a_1^\dagger - a_2^\dagger\right)^2\right\} |00\rangle \tag{46}$$

or

$$S'|00\rangle = \mathrm{e}^{-\frac{\lambda}{2}} \int \frac{\mathrm{d}^2\eta}{\pi} |\eta\rangle\, \mathrm{e}^{-(\mathrm{e}^{-2\lambda}\eta_1^2 + \eta_2^2)/2}. \tag{47}$$

Moreover, $S'(P_1 - P_2)S'^{-1} = \mathrm{e}^{\lambda}(P_1 - P_2)$, $S'(Q_1 - Q_2)S'^{-1} = \mathrm{e}^{-\lambda}(Q_1 + Q_2)$. Due to

$$[(P_1 + P_2)(Q_1 + Q_2), (Q_1 - Q_2)(P_1 - P_2)] = 0$$

we see

$$\langle\eta_1, \eta_2|\, \mathrm{e}^{\mathrm{i}\lambda[(P_1+P_2)(Q_1+Q_2)-(Q_1-Q_2)(P_1-P_2)]/2-\lambda} = \langle\eta_1, \eta_2|\, \mathrm{e}^{\mathrm{i}\lambda[P_2Q_1+Q_2P_1]} = \mathrm{e}^{-\lambda}\langle\mathrm{e}^{-\lambda}\eta_1, \mathrm{e}^{-\lambda}\eta_2| \tag{48}$$

which is consistent with equations (7) and (12).

6. Wigner distribution functions of the one-sided squeezed state

The Wigner distribution is a useful concept in quantum mechanics, quantum statistics, quantum optics and tomography technique [14–20]. Here we derive the Wigner functions of the one-sided squeezed states. The two-mode Wigner operator in the $\langle\eta|$ representation was obtained in [21]:

$$\Delta(\sigma, \gamma) = \int \frac{\mathrm{d}^2\eta}{\pi^3} |\sigma - \eta\rangle\langle\sigma + \eta|\, \mathrm{e}^{\eta\gamma^* - \eta^*\gamma}. \tag{49}$$

Because when we perform this integration with the IWOP technique and identify

$$\gamma = \alpha + \beta^* \qquad \sigma = \alpha - \beta^* \tag{50}$$

we see that it is just equal to the direct product of two single-mode Wigner operators

$$\Delta(\sigma, \gamma) = \Delta_1(\alpha, \alpha^*) \otimes \Delta_2(\beta^*, \beta) \tag{51}$$

where [22]

$$\Delta_1(\alpha_1, \alpha_1^*) = \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{2\pi}\, \mathrm{e}^{\mathrm{i}p_1u} \left|q_1 + \frac{u}{2}\right\rangle_{11}\left\langle q_1 - \frac{u}{2}\right| = \pi^{-1} : \mathrm{e}^{-2(\alpha^* - a_1^\dagger)(\alpha - a_1)} : \quad \alpha = \frac{q_1 + \mathrm{i}p_1}{\sqrt{2}} \\ \Delta_2(\beta^*, \beta) = \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{2\pi}\, \mathrm{e}^{\mathrm{i}p_2u} \left|q_2 + \frac{u}{2}\right\rangle_{22}\left\langle q_2 - \frac{u}{2}\right| = \pi^{-1} : \mathrm{e}^{-2(\beta^* - a_2^\dagger)(\beta - a_2)} : \quad \beta = \frac{q_2 + \mathrm{i}p_2}{\sqrt{2}}. \tag{52}$$

Further, performing the integration of $\Delta\,(\sigma, \gamma)$ over $\mathrm{d}^2\gamma$ leads to a projection operator

$$\int \mathrm{d}^2\gamma\, \Delta(\sigma, \gamma) = \frac{1}{\pi}|\eta\rangle\langle\eta||_{\eta=\sigma}. \tag{53}$$

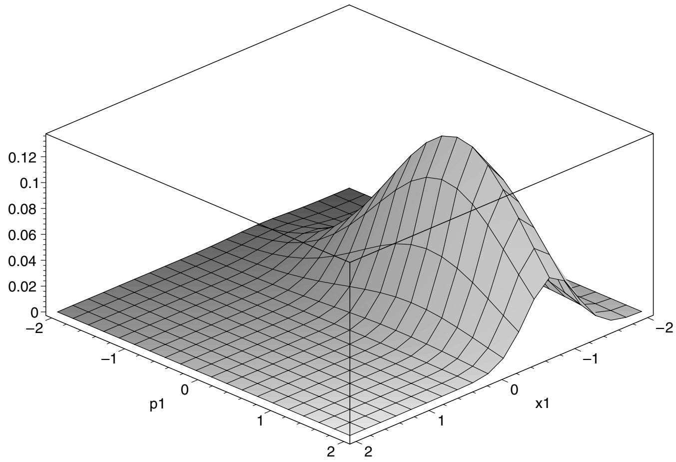

Figure 1. Plot of the Wigner function defined in equation (56), for $q_2 = p_2 = 1, q_1 = -2\ldots2,\ p_1 = -2\ldots2$, when $\lambda = 1$.

Thus, $\langle\psi|\int \mathrm{d}^2\gamma\ \Delta(\sigma,\gamma)|\psi\rangle = \frac{1}{\pi}|\langle\eta = \sigma|\psi\rangle|^2$ is a marginal distribution in the η variable. Similarly,

$$\langle\psi|\int \mathrm{d}^2\sigma\ \Delta(\sigma,\gamma)|\psi\rangle = \frac{1}{\pi}\langle\psi|\xi\rangle\langle\xi|\psi\rangle|_{\xi=\gamma} \tag{54}$$

is another marginal distribution in the ξ variable. Thus, the Wigner function $\langle\psi|\Delta(\sigma,\gamma)|\psi\rangle$ represents a distribution in the η–ξ phase space. Due to equations (11) and (49) we have the following matrix elements:

$$\langle\eta'|\Delta(\sigma,\gamma)|\eta\rangle = \frac{1}{\pi}\delta^{(2)}(\eta' + \eta - 2\sigma)\exp[(\eta-\sigma)\gamma^* + (\sigma^* - \eta^*)\gamma]. \tag{55}$$

From (31), (11) and (49) we deduce the Wigner function of the new squeezed state

$$\begin{aligned}\langle 00|S^\dagger\Delta(\sigma,\gamma)S|00\rangle &= \mathrm{e}^{\lambda}\int\frac{\mathrm{d}^2\eta'}{\pi}\int\frac{\mathrm{d}^2\eta}{\pi}\langle\eta'|\Delta(\sigma,\gamma)|\eta\rangle\,\mathrm{e}^{-(\eta_1'^2+\eta_2'^2\mathrm{e}^{-2\lambda}+\eta_1^2+\eta_2^2\mathrm{e}^{-2\lambda})/2}\\ &= \mathrm{e}^{\lambda}\int\frac{\mathrm{d}^2\eta'}{\pi}\int\frac{\mathrm{d}^2\eta}{\pi^2}\delta^{(2)}(\eta'+\eta-2\sigma)\\ &\quad\times\exp\left[(\eta-\sigma)\gamma^*+(\sigma^*-\eta^*)\gamma-\left(\eta_1'^2+\eta_2'^2\,\mathrm{e}^{-2\lambda}+\eta_1^2+\eta_2^2\,\mathrm{e}^{-2\lambda}\right)/2\right]\\ &= \frac{1}{\pi^2}\exp\left[-\gamma_2^2-\sigma_1^2-\mathrm{e}^{2\lambda}\gamma_1^2-\mathrm{e}^{-2\lambda}\sigma_2^2\right]\end{aligned} \tag{56}$$

where

$\gamma_1 = \frac{1}{\sqrt{2}}(q_1+q_2) \qquad \gamma_2 = \frac{1}{\sqrt{2}}(p_1-p_2) \qquad \sigma_1 = \frac{1}{\sqrt{2}}(q_1-q_2) \qquad \sigma_2 = \frac{1}{\sqrt{2}}(p_1+p_2).$

Equation (56) is different from the usual two-mode squeezed state's Wigner function (see the difference between figure 1 and figure 3; also see figure 2),

$$\begin{aligned}\langle 00|S''^\dagger\Delta(\sigma,\gamma)S''|00\rangle &= \frac{1}{\mu^2}\int\frac{\mathrm{d}^2\eta'}{\pi}\int\frac{\mathrm{d}^2\eta}{\pi}\langle\eta'/\mu|\Delta(\sigma,\gamma)|\eta/\mu\rangle\,\mathrm{e}^{-(|\eta|^2+|\eta'|^2)/2}\\ &= \frac{1}{\pi\mu^2}\int\frac{\mathrm{d}^2\eta'}{\pi}\int\frac{\mathrm{d}^2\eta}{\pi}\delta^{(2)}[(\eta'+\eta)/\mu-2\sigma]\\ &\quad\times\exp[-(|\eta|^2+|\eta'|^2)/2+(\eta/\mu-\sigma)\gamma^*+(\sigma^*-\eta^*/\mu)\gamma]\\ &= \frac{1}{\pi^2}\exp[-\mu^2|\sigma|^2-|\gamma|^2/\mu^2].\end{aligned} \tag{57}$$

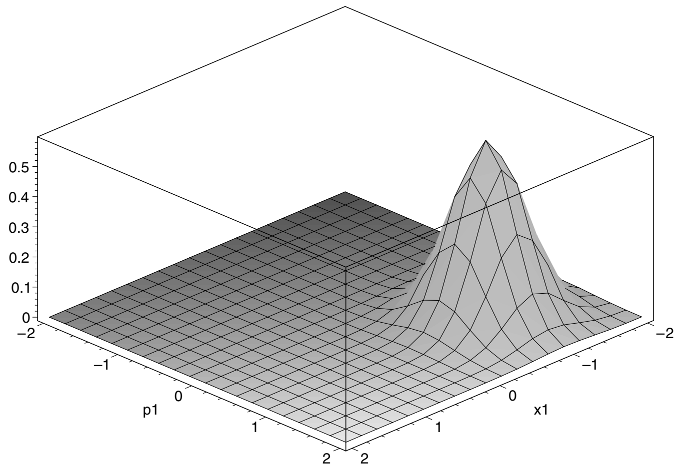

Figure 2. Plot of the Wigner function defined in equation (57), for $q_2 = p_2 = 1, q_1 = -2\ldots 2,\ p_1 = -2\ldots 2$, when $\lambda = 1$.

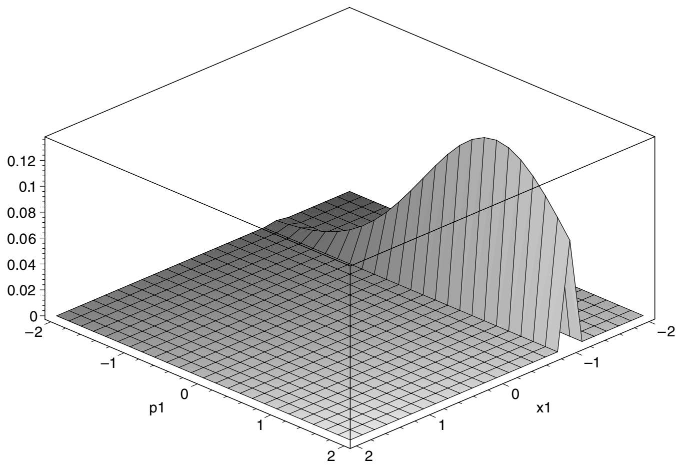

Figure 3. The variation of figure 1 when λ is changed to 10.

By analogy to the derivation of (56) we can obtain the Wigner function of $S'|00\rangle$

$$\langle 00|S'^{\dagger}\Delta(\sigma,\gamma)S'|00\rangle = \mathrm{e}^{-\lambda}\int\frac{\mathrm{d}^2\eta'}{\pi}\int\frac{\mathrm{d}^2\eta}{\pi}\langle\eta'|\Delta(\sigma,\gamma)|\eta\rangle\,\mathrm{e}^{-(\eta_2'^2+\eta_1'^2\mathrm{e}^{-2\lambda}+\eta_2^2+\eta_1^2\mathrm{e}^{-2\lambda})/2}$$
$$= \frac{1}{\pi^2}\exp\left[-\gamma_1^2-\sigma_2^2-\mathrm{e}^{2\lambda}\gamma_2^2-\mathrm{e}^{-2\lambda}\sigma_1^2\right]. \tag{58}$$

Hence one can recognize the one-sided squeezed states from their Wigner distributions.

7. Entangled state $|\eta\rangle$ as the basis for deriving a more general squeezed state

We have revealed the connection between the new squeezing operator (S, S') and the transform of the entangled state $|\eta\rangle$. Here we further consider two successive operations of $S(\mu)S'(\nu)$

on the vacuum state. Letting $\nu = \mathrm{e}^{\sigma}$, $\mu = \mathrm{e}^{\lambda}$ and using (31) and (44) we have

$$S(\mu)S'(\nu) = \mathrm{e}^{\mathrm{i}\frac{\lambda}{2}(P_1+P_2)(Q_1+Q_2)-\mathrm{i}\frac{\sigma}{2}(Q_1-Q_2)(P_1-P_2)-\frac{\lambda+\sigma}{2}}$$
$$= \sqrt{\mu\nu}\int \frac{\mathrm{d}^2\eta}{\pi}|\nu\eta_1, \mu\eta_2\rangle\langle\eta|. \tag{59}$$

Operating it on $|00\rangle$ yields a general squeezed state

$$S(\mu)S'(\nu)|00\rangle = \frac{2\sqrt{\mu\nu}}{\sqrt{L}}\exp\left\{\frac{1}{2L}\left[(\nu^2-\mu^2)\left(a_1^{\dagger 2}+a_2^{\dagger 2}\right)+2(1-\nu^2\mu^2)a_1^{\dagger}a_2^{\dagger}\right]\right\}|00\rangle \tag{60}$$

where $L = (1+\mu^2)(1+\nu^2)$. When $\nu = 1$, $L = (1+\mu^2)/2$, equation (60) reduces to (30). On the other hand, when $\mu = 1$, equation (60) reduces to (46). One can get a more general squeezing operator which corresponds to a more general transform $(\eta_1, \eta_2) \to \Lambda(\eta_1, \eta_2)$ in $|\eta\rangle$, by constructing the following ket-bra integration:

$$S(\Lambda) = (\det\Lambda)^{1/2}\int\frac{\mathrm{d}^2\eta}{\pi}\left|\Lambda\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\right| \tag{61}$$

where Λ is a 2×2 matrix which may be time dependent. Using the IWOP technique one can also perform the integration in (61) and get its normal product form. Generally speaking, equation (61) provides us with a way to design a two-mode squeezing. The preassigned general time-dependent transformations $\binom{\eta_1}{\eta_2} \to \Lambda\binom{\eta_1}{\eta_2}$ are taken as the starting point from which the time-evolution operator $U(t, 0)$ and then the corresponding time-dependent Hamiltonian are derived. This is similar to [23] where the design of squeezing by time-dependent harmonic oscillator is considered.

8. Application of the one-sided squeezed state

The new squeezed state $S|00\rangle$ may have potential applications in quantum information, for example, in entanglement swapping. Entanglement swapping is an approach for obtaining entanglement [24] which makes use of a projection of the state of two particles onto an entangled state. This projection measurement does not necessarily require a direct interaction between the two particles. For numerous uses of spatially separated entangled pairs of particles, each of the particles is entangled with one other partner particle, an appropriate measurement, for example, a Bell-state measurement of the partner particles, will automatically collapse the state of the remaining two particles into an entangled state. Through entanglement swapping one can entangle particles that do not even share any common past. Assuming that the initial state of four particles is

$$|\eta\rangle_{12}\otimes|\eta'\rangle_{34} \tag{62}$$

where particles 1 and 2 are entangled as $|\eta\rangle_{12}$, whereas particles 3 and 4 compose another state $|\eta'\rangle_{34}$.

We now perform a joint Bell-state measurement $S_{14}|00\rangle_{1414}\langle 00|S_{14}^{-1}$ on particles 1 and 4, where the two-mode squeezing operator S_{14} is defined in (26), then the state of particles 2 and 3, which originally are not correlated, will become entangled immediately after the measurement. To see this clearly, we calculate

$$\left({}_{14}\langle 00|S_{14}^{-1}\right)|\eta\rangle_{12}\otimes|\eta'\rangle_{34} = \mathrm{e}^{-\lambda/2}\int\frac{\mathrm{d}^2\eta''}{\pi}\,\mathrm{e}^{-(\eta_1''^2+\eta_2''^2\mathrm{e}^{-2\lambda})/2}{}_{14}\langle\eta''|\eta\rangle_{12}\otimes|\eta'\rangle_{34}. \tag{63}$$

Using (17) and the IWOP technique, we can prove

$${}_{14}\langle\eta''|\eta\rangle_{12}|\eta'\rangle_{34} = \exp\left\{-\tfrac{1}{2}(|\eta''|^2+|\eta|^2+|\eta'|^2)+\eta\eta''-\eta\eta'^*+\eta''^*\eta'^*\right\}$$
$$\times\exp\left[a_2^{\dagger}(\eta''-\eta^*-\eta'^*)-a_3^{\dagger}(\eta''^*-\eta-\eta')+a_2^{\dagger}a_3^{\dagger}\right]|00\rangle_{23}. \tag{64}$$

By comparison with the definition of EPR state (5), we see that (64) is just an entangled state of particles 2 and 3,

$$\left({}_{14}\langle\eta''|\right)|\eta\rangle_{12}|\eta'\rangle_{34} = A|\eta'' - \eta^* - \eta'^*\rangle_{23} \tag{65}$$

where

$$A = \exp\left[\tfrac{1}{2}(\eta\eta'' - \eta\eta'^* + \eta''^*\eta'^*) - \tfrac{1}{2}(\eta^*\eta''^* - \eta^*\eta' + \eta''\eta')\right] \tag{66}$$

is actually a phase factor, since it is the difference between a complex number and its complex conjugate. Substituting (65) into (63) gives

$$\left({}_{1,4}\langle 00|S_{1,4}^{-1}\right)|\eta\rangle_{1,2}|\eta'\rangle_{34} = \mathrm{e}^{-\lambda/2}A\int\frac{\mathrm{d}^2\eta''}{\pi}\,\mathrm{e}^{-(\eta_1''^2+\eta_2''^2\mathrm{e}^{-2\lambda})/2}|\eta'' - \eta^* - \eta'^*\rangle_{23} \tag{67}$$

From the definition expression (5) of $|\eta\rangle$, we know

$$|\eta\rangle_{23} = D(\eta)\,\mathrm{e}^{a_2^\dagger a_3^\dagger}|00\rangle_{23} \tag{68}$$

where

$$D(\eta) \equiv D\left(\eta a_2^\dagger - \eta^* a_2\right)$$

is a displacement operator. Using

$$D(\eta + \eta') = \exp\left[\tfrac{1}{2}(\eta^*\eta' - \eta'^*\eta)\right]D(\eta)D(\eta')$$

and (68), we obtain

$$|\eta'' - \eta^* - \eta'^*\rangle_{23} = \exp\left\{\tfrac{1}{2}[\eta''^*(\eta^* + \eta'^*) - (\eta + \eta')\eta'']\right\}D(-\eta^* - \eta'^*)|\eta''\rangle_{23}. \tag{69}$$

Substituting (69) into (67) and comparing the result with (31), we conclude that equation (67) denotes a displaced squeezed state $S_{23}|00\rangle_{23}$, thus particles 2 and 3 are entangled. The above derivation is direct and concise because we have fully used the $|\eta\rangle$ representation of the squeezing operators.

9. Generating a one-sided two-mode squeezed state by beam splitters

By comparing the form of (44) in the $\langle\eta|$ representation with the single-mode squeezing operator

$$\mathrm{e}^{-\lambda/2}\int_{-\infty}^{\infty}\mathrm{d}q\;|\mathrm{e}^{-\lambda}q\rangle_{11}\langle q| = \exp\left[\left(a_1^2 - a_1^{\dagger 2}\right)\lambda/2\right]$$

in the coordinate ${}_1\langle q|$ representation we naturally think of the following experiment: a single-mode squeezed state and a vacuum state overlapping on a beam splitter may produce a one-sided two-mode squeezed state at the output. The beam splitter operator is [25–27]

$$B = \exp\left[\frac{\theta}{2}\left(a_1^\dagger a_2 - a_1 a_2^\dagger\right)\right]. \tag{70}$$

It operates on the input state $\exp\left(-a_1^{\dagger 2}\tanh\lambda/2\right)|0\rangle_1 \otimes |0\rangle_2$ and yields

$$\begin{aligned} &B\operatorname{sech}^{1/2}\lambda\exp\left(-a_1^{\dagger 2}\tanh\lambda/2\right)|0\rangle_1 \otimes |0\rangle_2 \\ &\quad = \operatorname{sech}^{1/2}\lambda\exp\left[-\left(a_1^\dagger\cos\theta - a_2^\dagger\sin\theta\right)^2\tanh\lambda/2\right]|0,0\rangle. \end{aligned} \tag{71}$$

When $\theta = \pi/4$, a symmetric beam splitter case, (71) becomes

$$\operatorname{sech}^{1/2}\lambda\exp\left[-\frac{\tanh\lambda}{4}\left(a_1^\dagger - a_2^\dagger\right)^2\right]|0,0\rangle \tag{72}$$

which is just the one-sided squeezed state (46).

In summary, we have shown that the c-number asymmetric shrink (dilation) transform in the EPR entangled state, i.e. $|\eta_1, \eta_2\rangle \rightarrow |\eta_1, \eta_2/\mu\rangle$ (or $|\eta_1, \eta_2\rangle \rightarrow |\eta_1/\mu, \eta_2\rangle$), maps onto two kinds of one-sided two-mode squeezing operators in the Hilbert space. Equations (26), (36) and (44) are new relations between two-mode squeezing and the transforms of entangled states. The one-sided squeezing operators exhibit squeezing effect in the directions η_1 and η_2, respectively. The $\langle\eta|$ representation of the one-sided squeezing operators has enlightened us to overlap a single-mode squeezed state and a vacuum state on a 50/50 beam splitter to produce the one-sided two-mode squeezed state. The $\langle\eta|$ representation also makes the IWOP technique practical in deriving the normally ordered form of S and S' and in constructing the corresponding squeezed states. The entangled state representations of squeezing operators and the IWOP technique provide a way to derive preassigned squeezing evolution, thus is beneficial to squeezing design. The $\langle\eta|$ representation of squeezing operators is also useful for studying the phase behaviour of the Noh–Fougères–Mandel operational phase operator in the two-mode squeezed states as shown in section 3. The Wigner functions of the new squeezed states are also easily derived by virtue of the $|\eta\rangle$ representation. The more general two-mode squeezed states can be theoretically constructed using equation (61). Using the $\langle\eta|$ representation of the squeezing operators, the derivation of quantum swapping or quantum teleportation is direct and concise when the continuous EPR eigenstate or two-mode squeezed state are used as a quantum channel. The $\langle\eta|$ representation of the squeezing operators is a 'tie' connecting two-mode squeezing and quantum entanglement in a natural way.

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